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Hamiltonian Formulation of One-Chain and Two-Chain Systems

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We consider the Hamiltonian formulation of constrained dynamical systems with purely second-class constraints which flow from either one or two primary constraints, known as one-chain and two-chain systems, studied recently in detail by Mitra and Rajaraman, and quantize the theories using Dirac's procedure.

1. INTRODUCTION

Mitra and Rajaraman (1990*a*) have recently studied the constraint algebra for systems with purely second-class constraints which flow from either one or two primary constraints, known as one-chain and two-chain systems. For these systems, they derived general results for the algebra of Poisson brackets at the classical level which (in the absence of commutator anomalies) also hold for the commutators of the constraint operators in the corresponding quantized theories. They have also developed a general method by which it is sometimes possible to convert a classical dynamical theory with second-class constraints into a gauge-invariant theory with first-class constraints (Mitra and Rajaraman, 1990*b*), without any change in its physical content.

In this paper we consider the Hamiltonian formulation (Dirac, 1950, 1964) of one-chain and two-chain systems studied by Mitra and Rajaraman (1990*a*,*b*) and quantize both theories using Dirac's (1950, 1964) procedure. Some gauge-invariant reformulations of these theories have been considered by Mitra and Rajaraman (1990*b*).

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The one-chain and two-chain systems are considered in Sections 2 and 3, respectively. The summary and discussion is given in Section 4.

2. ONE-CHAIN SYSTEM

In this section we consider the Hamiltonian formulation (Dirac, 1950, 1964) of a one-chain system (Mitra and Rajaraman, 1990*a*,*b*) with purely second-class constraints which flow from only one primary constraint.

The Lagrangian (Mitra and Rajaraman, 1990*a*,*b*)

$$L=\frac{1}{2}\sum_{i=1}^{N-1}\left(\dot{q}_i+\sum_{j=1}^{N-1}c_{ij}q_j+d_iq_N\right)^2+q_N\sum_{i=1}^{N-1}b_iq_i \tag{2.1}$$

where q_i ($i=1,\ldots,N$) are canonical coordinates and $\dot{q}_i=dq_i/dt$ are the canonical velocities and b_i, d_i, and c_{ij} are some real constants, describes a family of models. For some special sets of values of the constants b_i, d_i, and c_{ij} the Lagrangian (2.1) could yield first-class constraints. However, for generic values [$b_i=\frac{1}{4}$, $d_i=-\frac{1}{2}$, $c_{ij}=\mathbb{1}_{4\times 4}$ (identity matrix); $i,j=1,\ldots,4$] it gives a theory with purely second-class constraints. We consider a specific example by taking $N=5$ as considered by Mitra and Rajaraman (1990*a*,*b*). Our Lagrangian for $N=5$ [Mitra and Rajaraman (1990*a*,*b*)] reads:

$$L_1=\frac{1}{2}(\dot{q}_1+q_1-\frac{1}{2}q_5)^2+\frac{1}{2}\sum_{i=2}^{4}(\dot{q}_i-\frac{1}{2}q_5)^2+\frac{1}{2}q_5\sum_{i=1}^{4}q_i \tag{2.2}$$

In the following, we consider the Hamiltonian formulation of the theory described by the Lagrangian L_1 equation (2.2), and quantize the system using Dirac's (1950, 1964) procedure.

By varying the action $S_1=\int L_1\,dt$ with respect to the coordinates q's we obtain the following Euler–Lagrange equations of motion:

$$q_1+\frac{1}{4}q_5-\ddot{q}_1+\frac{1}{2}\dot{q}_5=0 \tag{2.3a}$$

$$\frac{1}{4}q_5-\ddot{q}_2+\frac{1}{2}\dot{q}_5=0 \tag{2.3b}$$

$$\frac{1}{4}q_5-\ddot{q}_3+\frac{1}{2}\dot{q}_5=0 \tag{2.3c}$$

$$\frac{1}{4}q_5-\ddot{q}_4+\frac{1}{2}\dot{q}_5=0 \tag{2.3d}$$

$$-\frac{1}{2}(\dot{q}_1+q_1-\frac{1}{2}q_5)-\frac{1}{2}\sum_{i=2}^{4}(\dot{q}_1-\frac{1}{2}q_5)+\frac{1}{4}\sum_{i=1}^{4}q_i=0 \tag{2.3e}$$

It is easy to see that the Euler–Lagrange equations do not furnish dynamics for the coordinate q_5, since the acceleration $\ddot{q}_5$ does not appear in the equations. For considering the Hamiltonian formulation, we calculate the momenta ($p_i := \partial L_1/\partial \dot{q}_i$):

$$p_1 = \dot{q}_1 + q_1 - \tfrac{1}{2} q_5 \tag{2.4a}$$

$$p_2 = \dot{q}_2 - \tfrac{1}{2} q_5 \tag{2.4b}$$

$$p_3 = \dot{q}_3 - \tfrac{1}{2} q_5 \tag{2.4c}$$

$$p_4 = \dot{q}_4 - \tfrac{1}{2} q_5 \tag{2.4d}$$

$$p_5 = 0 \tag{2.4e}$$

We find that the momentum conjugate to q_5 is zero (since L_1 does not involve $\dot{q}_5$), and this implies that the theory possesses one primary constraint

$$\chi_1 := p_5 \approx 0 \tag{2.5}$$

The symbol $\approx$ expresses the "weak equality" in the sense of Dirac (1950, 1964). The canonical Hamiltonian H_{1c} is

$$H_{1c} = \tfrac{1}{2} \sum_{i=1}^{4} p_i^2 - p_1 q_1 + q_5 \sum_{i=1}^{4} (\tfrac{1}{2} p_i - \tfrac{1}{4} q_i) \tag{2.6}$$

After including the primary constraint χ_1 of the theory in the canonical Hamiltonian H_{1c} with the help of the as-yet-undetermined Lagrange multiplier v, we can write the total Hamiltonian H_{1T} (Dirac, 1950, 1964) as

$$H_{1T} = H_{1c} + P_5 v \tag{2.7}$$

For the Poisson bracket $\{\cdot, \cdot\}$ of two functions A and B, we choose the convention

$$\{A, B\} := \sum_{\alpha} \left(\frac{\partial A}{\partial q_\alpha} \frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha} \frac{\partial B}{\partial q_\alpha} \right) \tag{2.8}$$

Following Dirac's (1950, 1964) procedure, one finds that the system described by the Lagrangian L_1 of (2.2) has the following five secondary constraints (Mitra and Rajaraman, 1990*a*):

$$\chi_2 = \tfrac{1}{4} \sum_{i=1}^{4} (q_i - 2p_i) \approx 0 \tag{2.9a}$$

$$\chi_3 = \tfrac{1}{4} \sum_{i=1}^{4} p_i - \tfrac{1}{4}(q_1 + 2p_1) \approx 0 \tag{2.9b}$$

$$\chi_4 = \tfrac{1}{4}(q_1 - 2p_1) \approx 0 \tag{2.9c}$$

$$\chi_5 = -\tfrac{1}{4}(p_1 + q_1) \approx 0 \tag{2.9d}$$

$$\chi_6 = \tfrac{1}{4}(q_1 - 2p_1) - \tfrac{3}{16} q_5 \approx 0 \tag{2.9e}$$

The above secondary constraints have emerged from the single primary constraint χ_1. Thus the theory described by the Lagrangian L_1 equation (2.2) represents a one-chain system (Mitra and Rajaraman, 1990*a*). The Hamilton equations of motion are

$$\dot{q}_j = \frac{\partial H_{1T}}{\partial p_j} = [(p_1 - q_1 + \tfrac{1}{2}q_5), (p_2 + \tfrac{1}{2}q_5), (p_3 + \tfrac{1}{2}q_5), (p_4 + \tfrac{1}{2}q_5), v]; \qquad j = 1, 2, 3, 4, 5 \tag{2.10a}$$

$$-\dot{p}_j = \frac{\partial H_{1T}}{\partial q_j} = [(-p_1 - \tfrac{1}{4}q_5), (-\tfrac{1}{4}q_5), (-\tfrac{1}{4}q_5), (-\tfrac{1}{4}q_5), \sum_{i=1}^{4} (\tfrac{1}{2}p_i - \tfrac{1}{4}q_i)]; \qquad j = 1, 2, 3, 4, 5 \tag{2.10b}$$

We now calculate the Poisson brackets among the constraints χ's and obtain the matrix $M_{\alpha\beta} := \{\chi_\alpha, \chi_\beta\}$ as

$$M_{\alpha\beta} = \frac{3}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \tag{2.11}$$

with the inverse

$$M^{-1}_{\alpha\beta} = \frac{16}{3}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.12}$$

The constraints, by definition (Dirac, 1950, 1964) are second class if the matrix of the Poisson brackets among the constraints is invertible, and accordingly the contraints χ's are second class in the terminology of Dirac (1950, 1964), and consequently the Dirac quantization procedure is applicable to our theory. The Dirac bracket $\{\cdot, \cdot\}_D$ of two functions A and B is defined as

$$\{A, B\}_D = \{A, B\} - \sum_{\alpha,\beta} \{A, \Gamma_\alpha\}(\Delta^{-1}_{\alpha\beta})\{\Gamma_\beta, B\} \tag{2.13}$$

where the Γ's are the constraints of the theory, and $\Delta_{\alpha\beta}$ $(:= \{\Gamma_\alpha, \Gamma_\beta\})$ is the matrix of the Poisson brackets of the constraints Γ's.

The transition to quantum mechanics is made in the standard way by the replacement of the Dirac brackets by the operator commutation relations $[\cdot, \cdot]$, according to the Dirac (1950, 1964) quantization rule

$$\{A, B\}_D \to (-i)[\tilde{A}, \tilde{B}]; \qquad i=\sqrt{-1} \tag{2.14}$$

where A and B are any classical dynamical variables, such as q's and p's whereas $\tilde{A}$ and $\tilde{B}$ are the corresponding quantum mechanical operators on some Hilbert space.

Finally, the nonvanishing quantum commutators of the theory are

$$[\tilde{q}_j, \tilde{p}_j] = \tfrac{2}{3}i; \qquad j=2, 3, 4 \tag{2.15}$$

$$[\tilde{q}_j, \tilde{p}_k] = -\tfrac{1}{3}i; \qquad j\neq k, \quad j, k=2, 3, 4 \tag{2.16}$$

where the $\tilde{q}$'s and $\tilde{p}$'s are the quantum mechanical operators (on some Hilbert space) corresponding to the q's and p's, respectively.

3. TWO-CHAIN SYSTEM

In this section we consider the Hamiltonian formulation for the constrained dynamics of a two-chain system (Mitra and Rajaraman, 1990*a*,*b*) with purely second-class constraints which flow from two primary constraints. The Lagrangian of the theory (Mitra and Rajaraman, 1990*a*,*b*) reads

$$L_2 = \tfrac{1}{2}(\dot{q}_1 + q_5)^2 + \tfrac{1}{2}(\dot{q}_2 + q_6)^2 + \tfrac{1}{2}(\dot{q}_3^2 + \dot{q}_4^2) - q_5(q_2 + V_1(q_3, q_4)) + q_6(q_1 + V_2(q_3, q_4)) - V_3(q_3, q_4) \tag{3.1}$$

In the following we consider the Hamiltonian formulation of the theory described by the Lagrangian L_2 equation (3.1) and quantize the system using again Dirac's (1950, 1964) procedure.

The Euler–Lagrange equations of motion for the Lagrangian L_2 equation (3.1) are obtained by varying the action $S_2 = \int L_2\, dt$ with respect to the coordinates q's as follows:

$$q_6 - \ddot{q}_1 - \dot{q}_5 = 0 \tag{3.2a}$$

$$-q_5 - \ddot{q}_2 - \dot{q}_6 = 0 \tag{3.2b}$$

$$-q_5\, \partial_3 V_1 + q_6\, \partial_3 V_2 - \partial_3 V_3 - \ddot{q}_3 = 0 \tag{3.2c}$$

$$-q_5\, \partial_4 V_1 + q_6\, \partial_4 V_2 - \partial_4 V_3 - \ddot{q}_4 = 0 \tag{3.2d}$$

$$-(q_2 + V_1) = 0 \tag{3.2e}$$

$$-(q_1 + V_2) = 0 \tag{3.2f}$$

with

$$\partial_i = \frac{\partial}{\partial q_i}; \qquad \partial_i^2 = \frac{\partial^2}{\partial q_i^2}; \qquad \text{etc.} \tag{3.3}$$

One can see that the Euler–Lagrange equations do not furnish dynamics for the coordinates q_5 and q_6 since the accelerations $\ddot{q}_5$ and $\ddot{q}_6$ do not appear in the equations.

For considering the Hamiltonian formulation we need the momenta

$$p_1 = \dot{q}_1 + q_5 \tag{3.4a}$$

$$p_2 = \dot{q}_2 + q_6 \tag{3.4b}$$

$$p_3 = \dot{q}_3 \tag{3.4c}$$

$$p_4 = \dot{q}_4 \tag{3.4d}$$

$$p_5 = 0 \tag{3.4e}$$

$$p_6 = 0 \tag{3.4f}$$

We find that the momenta conjugate to q_5 and q_6 are zero (since L_2 does not involve $\dot{q}_5$ and $\dot{q}_6$), and this implies that the theory possesses two primary constraints.

$$\phi_1 := p_5 \approx 0 \tag{3.5a}$$

$$\psi_1 := p_6 \approx 0 \tag{3.5b}$$

The canonical Hamiltonian H_{2c} is

$$H_{2c} = \tfrac{1}{2} \sum_{i=1}^{4} p_i^2 - q_5(p_1 - q_2 - V_1) - q_6(p_2 + q_1 + V_2) + V_3 \tag{3.6}$$

After including the primary constraints ϕ_1 and ψ_1 of the theory in the canonical Hamiltonian H_{2c} with the help of the as-yet-undetermined Lagrange multipliers u and w, one can write the total Hamiltonian H_{2T} (Dirac 1950, 1964) as

$$H_{2T} = H_{2c} + p_5 u + p_6 w \tag{3.7}$$

Demanding that the primary constraints ϕ_1 and ψ_1 be preserved in the course of time, we obtain the secondary constraints

$$\phi_2 = (p_1 - q_2 - V_1) \approx 0 \tag{3.8a}$$

$$\psi_2 = (p_2 + q_1 + V_2) \approx 0 \tag{3.8b}$$

By demanding further that the secondary constraints ϕ_2 and ψ_2 be also preserved in the course of time, we obtain

$$\phi_3=(2q_6-p_2-p_3\,\partial_3V_1-p_4\,\partial_4V_1)\approx 0 \tag{3.9a}$$

$$\psi_3=(-2q_5+p_1+p_3\,\partial_3V_2+p_4\,\partial_4V_2)\approx 0 \tag{3.9b}$$

Demanding further that the constraints ϕ_3 and ψ_3 be also preserved in the course of time, one does *not* obtain any additional constraints, implying that the theory possesses only six constraints, ϕ_1, ψ_1, ϕ_2, ψ_2, and ϕ_3, ψ_3. Here, the first two constraints, namely, ϕ_1 and ψ_1, are primary and the rest are secondary and have emerged from the two primary constraints ϕ_1 and ψ_1. Accordingly, the theory described by the Lagrangian L_2 equation (3.1) represents a two-chain system. The Hamilton equations of motion are

$$\dot{q}_j=\frac{\partial H_{2T}}{\partial p_j}=[(p_1-q_5),\ (p_2-q_6),\ (p_3),\ (p_4),\ (u),\ (w)]$$

$$j=1,2,3,4,5,6 \tag{3.10a}$$

$$-\dot{p}_j=\frac{\partial H_{2T}}{\partial q_j}=[(-q_6),\ (q_5),\ (q_5\,\partial_3V_1-q_6\,\partial_3V_2+\partial_3V_3),$$

$$(q_5\,\partial_4V_1-q_6\,\partial_4V_2+\partial_4V_3),\ (-p_1+q_2+V_1),$$

$$(-p_2-q_1-V_2)];\qquad j=1,2,3,4,5,6 \tag{3.10b}$$

Next we calculate the Poisson brackets among the constraints ϕ's and ψ's [given by equations (3.5), (3.8), (3.9)]. In the case of the one-chain system considered in the previous section, we saw that the upper half subset of the constraint set had zero mutual Poisson brackets. For the present two-chain system it is also possible to have the upper half subset of the constraint set to have zero mutual Poisson brackets. In order for the upper half subset of the constraint set to have zero mutual Poisson brackets for the two-chain system under consideration we rename the constraints of the theory as follows:

$$\Omega_1=\phi_1=p_5\approx 0 \tag{3.11a}$$

$$\Omega_2=\psi_1=p_6\approx 0 \tag{3.11b}$$

$$\Omega_3=\phi_2=(p_1-q_2-V_1)\approx 0 \tag{3.11c}$$

$$\Omega_4=\psi_2=(p_2+q_1+V_2)\approx 0 \tag{3.11d}$$

$$\Omega_5=\phi_3=(2q_6-p_2-p_3\,\partial_3V_1-p_4\,\partial_4V_1)\approx 0 \tag{3.11e}$$

$$\Omega_6=\psi_3=(-2q_5+p_1+p_3\,\partial_3V_2+p_4\,\partial_4V_2)\approx 0 \tag{3.11f}$$

The matrix $R_{\alpha\beta} := \{\Omega_\alpha, \Omega_\beta\}$ for the Poisson brackets among the constraints Ω equation (3.11) is then obtained as

$$R_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & a & -b \\ 0 & 0 & 2 & 0 & -b & f \\ 0 & 2 & -a & b & 0 & d \\ -2 & 0 & b & -f & -d & 0 \end{bmatrix} \tag{3.12}$$

where the values of a, b, d, and f are given by

$$a = 1 + \partial_3^2 V_1 + \partial_4^2 V_1 \tag{3.13a}$$

$$b = \partial_3 \mathbf{V}_1\, \partial_3 \mathbf{V}_2 + \partial_4 \mathbf{V}_1\, \partial_4 \mathbf{V}_2 \tag{3.13b}$$

$$d = -p_3\, \partial_3^2 V_1\, \partial_3 V_2 - p_4\, \partial_4^2 V_1\, \partial_4 V_2 + p_3\, \partial_3 V_1\, \partial_3^2 V_2 + p_4\, \partial_4 V_1\, \partial_4^2 V_2 \tag{3.13c}$$

$$f = 1 + \partial_3^2 V_2 + \partial_4^2 V_2 \tag{3.13d}$$

The inverse of the matrix $R_{\alpha\beta}$ is

$$R_{\alpha\beta}^{-1} = \begin{bmatrix} 0 & g & f/4 & b/4 & 0 & -1/2 \\ -g & 0 & b/4 & a/4 & 1/2 & 0 \\ -f/4 & -b/4 & 0 & 1/2 & 0 & 0 \\ -b/4 & -a/4 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.14}$$

where

$$g = \frac{d}{4} - \frac{b^2}{8} + \frac{af}{8} \tag{3.15}$$

We thus find that the constraints Ω equation (3.11) are second class in the terminology of Dirac (1950, 1964), and consequently, Dirac's procedure is again applicable for the quantization of the two-chain system under consideration.

Finally, the nonvanishing quantum commutators of the theory are obtained as

$$[\tilde{p}_1, \tilde{p}_2] = i\left[g + \frac{b}{4} + \tfrac{1}{2}\right] \tag{3.16}$$

$$[\tilde{p}_1, \tilde{p}_3] = i\left[g\, \partial_3 V_1 + \frac{f}{4} p_3\, \partial_3^2 V_1 + \tfrac{1}{2} p_3\, \partial_3^2 V_2 + \tfrac{1}{2}\, \partial_3 V_1\right] \tag{3.17}$$

$$[\tilde{p}_1, \tilde{p}_4] = i\left[g\ \partial_4 V_1 + \frac{f}{4} p_4\ \partial_4^2 V_1 + \tfrac{1}{2} p_4\ \partial_4^2 V_2 + \tfrac{1}{2}\ \partial_4 V_1\right] \tag{3.18}$$

$$[\tilde{p}_2, \tilde{p}_3] = i\left[\frac{b}{4} p_3\ \partial_3^2 V_1 - \tfrac{1}{2}\ \partial_3 V_2 - \frac{a}{4}\ \partial_3 V_1 - \tfrac{1}{2} p_3\ \partial_3^2 V_1\right] \tag{3.19}$$

$$[\tilde{p}_2, \tilde{p}_4] = i\left[\frac{b}{4} p_4\ \partial_4^2 V_1 - \tfrac{1}{2}\ \partial_4 V_2 - \frac{a}{4}\ \partial_4 V_1 - \tfrac{1}{2} p_4\ \partial_4^2 V_1\right] \tag{3.20}$$

$$[\tilde{p}_3, \tilde{p}_4] = i\left[\frac{b}{4} p_4\ \partial_3 V_1\ \partial_4^2 V_1 - \tfrac{1}{2}\ \partial_3 V_1\ \partial_4 V_2 - \frac{b}{4} p_3\ \partial_4 V_1\ \partial_3^2 V_1 + \tfrac{1}{2}\ \partial_3 V_2\ \partial_4 V_1\right] \tag{3.21}$$

$$[\tilde{p}_j, \tilde{p}_6] = i\left[-\frac{f}{2}, \left(2 - \frac{b}{2}\right), \left(-\frac{b}{2}\ \partial_3 V_1\right), \left(-\frac{b}{2}\ \partial_4 V_1\right)\right] \quad j = 1, 2, 3, 4 \tag{3.22}$$

$$[\tilde{q}_1, \tilde{q}_j] = i\left[\left(\frac{1}{2} - \frac{b}{4}\right), \left(-\frac{b}{4}\ \partial_3 V_1\right), \left(-\frac{b}{4}\ \partial_4 V_1\right), \left(\tfrac{1}{2} - g\right), \left(\frac{a}{4}\right)\right] \quad j = 2, 3, 4, 5, 6 \tag{3.23}$$

$$[\tilde{q}_j, \tilde{q}_6] = i\left[\left(-\tfrac{1}{2}\right), \left(-\tfrac{1}{2}\ \partial_3 V_1\right), \left(-\tfrac{1}{2}\ \partial_4 V_1\right), \left(\frac{b}{4}\right)\right] \quad j = 2, 3, 4, 5 \tag{3.24}$$

$$[\tilde{q}_j, \tilde{q}_5] = i\left[\left(\frac{f}{4}\right), \left(\frac{f}{4}\ \partial_3 V_1 + \tfrac{1}{2}\ \partial_3 V_2\right), \left(\frac{f}{4}\ \partial_4 V_1 + \tfrac{1}{2}\ \partial_4 V_2\right)\right] \quad j = 2, 3, 4 \tag{3.25}$$

$$[\tilde{p}_j, \tilde{q}_j] = i\left[(-1 + g), \left(-\frac{b}{4}\right), \left(-1 - \frac{b}{4}\ (\partial_3 V_1)^2\right), \left(-1 - \frac{b}{4}\ (\partial_4 V_1)^2\right), (-2)\right]; \qquad j = 1, 2, 3, 4, 6 \tag{3.26}$$

$$[\tilde{p}_j, \tilde{q}_1] = i\left[\left(-\frac{a}{4}\right), \left(\tfrac{1}{2}\,\partial_3 V_2 - \frac{b}{4} p_3\, \partial_3^2 V_1\right),\right.$$
$$\left.\left(\tfrac{1}{2}\,\partial_4 V_2 - \frac{b}{4} p_4\, \partial_4^2 V_1\right), \left(\frac{b}{2}\right)\right]; \qquad j = 2, 3, 4, 6 \tag{3.27}$$

$$[\tilde{p}_j, \tilde{q}_2] = i\left[\left(-\frac{f}{4}\right), \left(\left(\frac{1}{2} - \frac{b}{4}\right)\partial_3 V_1\right), \left(\left(\frac{1}{2} - \frac{b}{4}\right)\partial_4 V_1\right)\right]; \qquad j = 1, 3, 4 \tag{3.28}$$

$$[\tilde{p}_j, \tilde{q}_3] = i\left[\left(-\frac{f}{4}\,\partial_3 V_1 - \tfrac{1}{2}\,\partial_3 V_2\right), \left(\left(\frac{1}{2} - \frac{b}{4}\right)\partial_3 V_1\right),\right.$$
$$\left.\left(-\frac{b}{4}\,\partial_4 V_1\, \partial_3 V_1\right)\right]; \qquad j = 1, 2, 4 \tag{3.29}$$

$$[\tilde{p}_j, \tilde{q}_4] = i\left[\left(-\frac{f}{4}\,\partial_4 V_1 - \tfrac{1}{2}\,\partial_4 V_2\right), \left(\left(\frac{1}{2} - \frac{b}{4}\right)\partial_4 V_1\right),\right.$$
$$\left.\left(-\frac{b}{4}\,\partial_3 V_1\, \partial_4 V_1\right)\right]; \qquad j = 1, 2, 3 \tag{3.30}$$

$$[\tilde{p}_j, \tilde{q}_6] = i\left[\left(\frac{b}{4}\right), \left(\frac{a}{4}\right), \left(\frac{a}{4}\,\partial_3 V_1 - \tfrac{1}{2} p_3\, \partial_3^2 V_1\right),\right.$$
$$\left.\left(\frac{a}{4}\,\partial_4 V_1 + \tfrac{1}{2} p_4\, \partial_4^2 V_1\right)\right]; \qquad j = 1, 2, 3, 4 \tag{3.31}$$

$$[\tilde{p}_j, \tilde{q}_5] = i\left[-\left(g + \frac{b}{4}\right), \left(\frac{f}{2}\right)\right]; \qquad j = 2, 6 \tag{3.32}$$

$$[\tilde{p}_3, \tilde{q}_5] = i\left[-g\,\partial_3 V_1 - \frac{f}{4} p_3\, \partial_3^2 V_1 - \tfrac{1}{2} p_3\, \partial_3^2 V_2\right] \tag{3.33}$$

$$[\tilde{p}_4, \tilde{q}_5] = i\left[-g\,\partial_4 V_1 - \frac{f}{4} p_4\, \partial_4^2 V_1 - \tfrac{1}{2} p_4\, \partial_4^2 V_2\right] \tag{3.34}$$

where the $\tilde{q}$'s and $\tilde{p}$'s are again the quantum mechanical operators corresponding to the q's and p's, respectively.

4. SUMMARY AND DISCUSSION

As mentioned in Section 1, the results derived by Mitra and Rajaraman (1990*a*) for the algebra of Poisson brackets (at the classical level) for the one-chain and two-chain systems considered in Sections 2 and 3 also hold for the commutators of the constraint operators in the corresponding quantized theories (Mitra and Rajaraman, 1990*a*).

The fact that the constraints associated with the Lagrangians L_1 equation (2.2) and L_2 equation (3.1) are second class is due to the lack of gauge invariance of L_1 and L_2. Gauge symmetry, however, when present in a theory has many beneficial consequences. Following the method of Mitra and Rajaraman (1990*b*), it is possible to reformulate the theories L_1 equation (2.2) and L_2 equation (3.1) as gauge-invariant theories (i.e., theories with first-class constraints) without any change in their physical contents.

The main idea lies in suitably modifying the canonical Hamiltonian (and correspondingly the Lagrangian) of a particular (given) second-class (or gauge-noninvariant) theory in such a way that the chain of the constraints is terminated at a desired point before its natural end. The constraints of the modified theory then form a set of first-class constraints and consequently the resulting modified theory becomes a gauge-invariant theory. The secondary constraints which did not appear in the modified theory (but were otherwise present in the original second-class theory) can now be imposed on the modified (gauge-invariant) theory as gauge-fixing conditions, so that the total set of constraints again becomes a second-class set. The Dirac quantization of the modified (gauge-invariant) theory under such gauge-fixing conditions remains identical with that of the original second-class (gauge-noninvariant) theory. Consequently, the physical content of the modified gauge-invariant theory under such gauge-fixing conditions remains the same as that of the original second-class theory. The physical equivalence of the modified and the original theory is therefore transparent. Some possible gauge-invariant reformulations of the theories for L_1 equation (2.2) and L_2 equation (3.1) (considered in Sections 2 and 3) have been given explicitly by Mitra and Rajaraman (1990*b*), for the details of which we refer to their work. The details of the properties of the algebra of Poisson brackets for the theories L_1 equation (2.2) and L_2 equation (3.1) are given by Mitra and Rajaraman (1990*a*). The properties of the algebra of Dirac brackets and those of the algebra of commutators for both the theories L_1 equation (2.2) and L_2 equation (3.1) are the same as the properties of the algebra of Poisson brackets for these theories derived by Mitra and Rajaraman (1990*a*) because there are no commutator anomalies in these theories (Mitra and Rajaraman, 1990*a*).

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